

Stress energy tensor renormalization for a spherically symmetric massive scalar field on a quantum space-time

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We consider a massive scalar field living on the recently found exact quantum space-time corresponding to vacuum spherically symmetric loop quantum gravity. The discreteness of the quantum space time naturally regularizes the scalar field, eliminating divergences. However, the resulting finite theory depends on the details of the micro physics. We argue that such dependence can be eliminated through a finite renormalization and discuss its nature. This is an example of how quantum field theories on quantum space times deal with the issues of divergences in quantum field theories.

Keywords: Stress tensor; renormalization.

The solution to the constraint equations of loop quantum gravity in the context of vacuum spherically symmetric space-times has been found in closed form in loop quantum gravity¹. The resulting quantum space-time has a discrete nature. It is characterized by the ADM mass of the space-time, a one dimensional spin network given by a graph g and a tower of integers corresponding to the valences of the spin network \vec{k} . The latter are related to the eigenvalues of the triad in the radial direction \hat{E}^x . This quantity in turn determines the areas of the spheres of symmetry which are therefore quantized. As a consequence the spacing of the spin network is bounded below by $\ell_{\text{Planck}}^2/(2r)$. For macroscopic situations the spacing is therefore very small, in fact sub-Planckian.

Quantum fields have been considered on such a space-time. Hawking radiation has been derived with small corrections² and the Casimir effect between two spherical shells has been considered³. The main difference between considering a quantum field on a quantum space-time as opposed to a classical space-time is that the field equations become discretized and the divergences naturally regulated as was anticipated in⁴. Because the spacing of the lattice is very fine, the resulting discrete equations can be well approximated by the equations of the continuum. However, because the lattice spacing is related to the Planck scale, quantities that would have diverged in the continuum are finite but large, leaving an unacceptable

imprint of the micro physics on the macro physics. We argue that such dependence must be eliminated via a (finite) renormalization and the dependence on the micro physics can be absorbed in a redefinition of the bare constants of the theory. In this note we would like to sketch the resulting calculation. We will operate mostly in the continuum as an approximation to the correct discrete theory and introduce the discreteness in key steps where it is relevant.

We start from the action for spherically symmetric gravity with a possible $1 + 1$ dimensional cosmological constant,

$$S = \int dx^0 dr \frac{\sqrt{-g^{(2)}}}{4G_B} (Rr^2 - 2\Lambda_B) \quad (1)$$

where G_B and Λ_B are the bare values of Newton's constant and the cosmological constant.

We will consider the stress tensor,

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}}, \quad (2)$$

where

$$W = -\frac{i}{2} \text{Tr} (\ln (-G_F)), \quad (3)$$

with G_F the Feynman propagator for the massive scalar field,

$$(\nabla_\mu \nabla^\mu + m^2 + \xi R) G(x, x') = g^{-1/2}(x) \delta(x - x'). \quad (4)$$

Since we are interested in the short distance behavior of the Green's function we can expand the metric in Riemann normal coordinates around a point x' , we can expand the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^\alpha y^\beta - \frac{1}{6} R_{\mu\alpha\nu\beta,\gamma} y^\alpha y^\beta y^\gamma + \dots, \quad (5)$$

with $y = x - x'$, allowing us to expand the propagator as,

$$G(k) = (k^2 - m^2)^{-1} - \left(\frac{1}{6} - \xi\right) R (k^2 - m^2)^{-2} + \dots \quad (6)$$

If we now rescale the propagator $\bar{G} = (-g)^{1/4} G$ and we go to momentum space, and recall that we are in spherical symmetry so only the radial and time coordinates are involved, we get,

$$\begin{aligned} \bar{G}(x, x') &= \int \frac{d^2 k}{(2\pi)^2} \exp(-ik_0 y^0 + ik_1 y^1) \\ &\times \left[1 + a_1(x, x') \left(-\frac{\partial}{\partial m^2}\right) + a_2(x, x') \left(-\frac{\partial}{\partial m^2}\right)^2 \right] \frac{1}{k_0^2 - k_1^2 - m^2}, \end{aligned} \quad (7)$$

with,

$$a_1(x, x') = \left(\frac{1}{6} - \xi\right) R - \frac{1}{2} \left(\frac{1}{6} - \xi\right) R_{,\alpha} y^\alpha - \frac{1}{3} a_{\alpha\beta} y^\alpha y^\beta. \quad (8)$$

The quantity $a_{\alpha\beta}$ is a geometric expression involving linear and quadratic terms in the scalar curvature, Ricci and Riemann tensor. In the generic $3+1$ case the term involving a_2 also leads to divergent corrections that need to be compensated introducing counterterms quadratic in the curvature. In spherical symmetry one does not need to consider such term, as we shall see.

To compute the Green's function it is good to use the identity,

$$(k^2 - m^2)^{-1} = -i \int_0^\infty ds \exp(is(k^2 - m^2)), \quad (9)$$

which allows to integrate in k and yield,

$$\bar{G}(x, x') = -\frac{i}{4\pi} \int_0^\infty \frac{ds}{s} \exp\left(-im^2s + \frac{\sigma}{2is}\right) \left[1 + a_1(x, x')is + a_2(x, x')(is)^2\right], \quad (10)$$

where σ is related to the geodesic distance squared between x and x' , $\sigma = y^2/2$.

This is how the calculation on a classical background goes. However, the quantum background introduces a difference. As we argued, the condition of the quantization of the areas of symmetry leads to an effective quantization of the radial coordinate with $r_i^2 = \ell_{\text{Planck}}^2 k_i$ with i the label of the vertex of the spin network associated with the radial position r_i . We will consider the simplest case of a spin network that is equispaced in normal coordinates with lattice spacing Δ . This imposes a cutoff in the radial integral in k^1 of $2\pi/\Delta$ as is common on a lattice. Then the Green's function will take the form,

$$\begin{aligned} \bar{G}_\Delta(x, x') = & -\frac{i}{8\pi} \int_0^\infty \frac{ds}{s} \exp\left(-im^2s + \frac{\sigma}{2is}\right) \left[\text{erf}\left(\frac{\sqrt{i}}{2} \left(\frac{4\pi s - \Delta y^1}{\Delta\sqrt{s}}\right)\right) \right. \\ & \left. - \text{erf}\left(\frac{-\sqrt{i}}{2} \left(\frac{4\pi s + \Delta y^1}{\Delta\sqrt{s}}\right)\right) \right] \\ & \times \left[1 + a_1(x, x')is + a_2(x, x')(is)^2\right]. \end{aligned} \quad (11)$$

As a result the effective action is finite and takes the form (eq. 6.35 in⁵),

$$W = \frac{i}{2} \int dx^0 dr \sqrt{-g^{(2)}} \lim_{x' \rightarrow x} \int_{m^2}^\infty dm^2 \bar{G}_\Delta(x, x'). \quad (12)$$

From here we can identify the effective Lagrangian where we study the divergence,

$$L_{\text{effective}}^{\text{div}} = -\frac{i}{2} \int_0^\infty \frac{ds}{s^2} \frac{\exp(-im^2s)}{4\pi} \text{erf}\left(\frac{\sqrt{is}2\pi}{\Delta}\right) (1 + a_1is). \quad (13)$$

For the particular background quantum state we chose with an equispaced lattice with invariant distance among vertices of the spin network given by Δ , the first two terms in the expansion in powers of (is) would lead to divergent contributions in the limit $\Delta \rightarrow 0$. For a finite, sub-Planckian Δ they are very large. They can be

renormalized by absorbing them in the bare cosmological constant and bare Newton constant. The total gravitational Lagrangian density becomes,

$$L_{\text{ren}} = \left(-g^{(2)}\right)^{1/2} \left[- \left(A + \frac{\Lambda_B}{8\pi G_B} \right) + \left(B + \frac{r^2}{16\pi G_B} \right) R \right], \quad (14)$$

To compute A and B we need to compute the integrals in the effective Lagrangian (13). We have done this in two different ways: a) Using the ascending power series of the error function and computing the exact sum of the integrals of each term and analytically extending to the asymptotic region; b) Using the asymptotic series. In both cases the result is,

$$A = \frac{\pi}{\Delta^2} + \frac{m^2}{8\pi} \left[1 + \ln \left(\frac{16\pi^2}{m^2 \Delta^2} \right) \right], \quad (15)$$

$$B = \frac{1}{24\pi} \ln \left(\frac{4\pi}{m\Delta} \right), \quad (16)$$

and since we are taking into account only the spherical mode of the field, the cosmological constant term that arises is not a spherically symmetric reduction of four dimensional gravity, but it is the cosmological constant term one has in $1+1$ dimensional theories.

For the previous calculations, we have assumed that the geodesic distance between points of the lattice, which determines the possible values of k , is Δ and is constant for all points in the radial direction. This is not really required and the theory allows variable spacing. However, if one wishes to reabsorb the dependence on the microscopic states, this will require counterterms that are state-dependent or to choose privileged states to renormalize. An intriguing possibility would be to consider conformal gravity where the size of these distances can be chosen arbitrarily and therefore one can make the spacing uniform.

Summarizing, the use of quantum field theory in quantum space-time techniques naturally regularizes the divergences of the field theory, replacing the divergent terms with terms that are large and depend on the details of the micro-structure of the background quantum state. Such dependence can be removed by a (finite) renormalization that absorbs the dependence on the micro-structure in the coupling constants of the theory, in this particular model the cosmological constant and Newton's constant.

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